Certification for Consistent Quality Provision*

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Abstract

A competent firm trades with a sequence of consumers who are unsure about this firm's competence and effort to supply quality. I study the design of certification rating systems that provide this firm with incentives to consistently supply quality. I characterize necessary and sufficient conditions given which these rating systems are viable, and explicitly construct one such rating system. This system discloses the firm's competence upon sufficiently many consecutive good trade outcomes and hides all information otherwise. It illuminates the role of initial audit, coarse ratings, graduated punishments, and low certification standard in providing incentives for consistent quality provision.

Keywords: certification, ratings, reputations.

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1 Introduction

1.1 Ratings and reputations

In markets for experience goods, including hotels, restaurants, and various types of service providers, consumers are often unsure about the sellers' competence and efforts to supply quality. The resulting adverse selection and moral hazard undermine these sellers' incentives to supply quality and, in turn, social welfare. In these markets, certification rating systems are important social institutions. These systems provide consumers with information and thus enable sellers to build their reputations for competence, thereby sustaining these sellers' incentives to consistently supply quality. Examples of these rating systems abound. For instance, the ISO 9001 certification website writes that their certification helps ensure that customers get consistent, good-quality products and services.[1] These rating systems also play an increasingly important role in online labor markets such as *Upwork*.

The use of rating systems to provide reputational incentives for consistent quality provision must contend with a fundamental tension that has long been recognized by economists since Holmström (1982). To help sellers who supply quality to build their reputations, these systems must be sufficiently informative. But to prevent these sellers from building their reputations too successfully, who then rest on their laurels and shirk, these systems must also be sufficiently uninformative. Important questions arise. What are the circumstances in which rating systems that motivate consistent quality provision are viable, and what are the circumstances in which they are not? When these systems are viable, how can they be designed?

To address these questions, I study a model in which a firm exerts hidden effort to serve a sequence of consumers. Only the firm knows whether it is a competent type or an inept type. A competent type is tempted to shirk, because its effort is costly. An inept type only exerts zero effort, because its effort is prohibitively costly. In each trade, the consumer sees some recent ratings of the firm and pays the firm his perceived value of the trade outcome; this outcome is a noisy signal of the firm's

[1] See https://www.iso.org/iso-9001-quality-management.html, 20 February, 2020.

effort. The consumer's payment is higher if he expects higher effort from the firm. Ratings are published by a rating system, which is a function of the firm's past ratings and past outcomes. A certifier chooses *ex ante* a system that virtually implements full effort, i.e., a system that motivates full effort from a patient competent firm in virtually all trades in equilibrium.

1.2 Ratings for virtual full effort

My main result identifies necessary and sufficient conditions given which a rating system that virtually implements full effort exists, and explicitly constructs one such rating system.

These conditions formalize the tension that ratings must contend with in providing reputational incentives and afford sharp comparative statics. In an equilibrium in which the competent firm exerts full effort in virtually all trades, consumers' payments in these trades are proportional to this firm's reputations, namely consumers' beliefs that this firm is competent. To sustain this equilibrium, ratings must induce sufficiently high reputations to reward good trade outcomes and sufficiently low reputations to punish bad trade outcomes. As a result, these conditions are more stringent if outcomes are noisier signals of efforts, in which case the competent firm has weaker effort incentives. These conditions are also more stringent if the firm's prior reputation is higher; in this case, ratings are less effective in inducing low reputations for punishments. These conditions also call for limited rating records so that ratings are flexible enough to induce high reputations in some trades and low reputations in the other trades.

In constructing a rating system that virtually implements full effort, two difficulties must be addressed. One difficulty is to motivate the competent firm's full effort when it is enjoying a high reputation: if this firm produces a bad outcome, then ratings must punish this firm by convincing consumers that this firm is unlikely to be competent, i.e., by inducing low reputations, despite having just convinced them otherwise. Another difficulty is to motivate this firm's full effort when it is suffering from a low reputation: if this firm produces a bad outcome, then it must be further punished but its reputation must be, on average, at least its prior reputation given consumers' Bayesian inferences.

My construction uses a combination of dynamic screening and graduated punishments to overcome these difficulties. This system starts with a screening phase of finitely many periods, during which the firm's ratings are uninformative. When this phase concludes, a firm is said to be qualified if it has produced sufficiently many good outcomes and is said to be unqualified otherwise. In this phase, the competent firm exerts full effort in most trades to pursue a higher future payoff from qualification. A competent firm is almost certain to be qualified given its efforts in this phase, while an inept firm is almost certain to be unqualified.

When the screening phase concludes, a certification phase begins and lasts forever. In virtually all trades during this certification phase, a qualified competent firm is rated as "top" upon consecutively many good outcomes or as "normal" otherwise. The top rating rewards this firm by almost revealing its competence, i.e., by inducing a reputation that is close to one. The normal rating serves as punishment: when a consumer's rating observations include only the normal rating, this firm's reputation is approximately the lowest reputation that it can obtain on average, i.e., the prior reputation. Punishments are graduated: upon every bad outcome that this firm produces, it needs to produce at least one more good outcome to obtain the top rating. This threat of further-lengthening punishments motivates this firm's full effort when it is already suffering from a low reputation. This threat also motivates this firm's full effort at the top rating by ensuring that punishments upon a bad outcome at the top rating are sufficiently severe: because outcomes are noisy signals of efforts, after a certain number of trades that follow a bad outcome at the top rating, entering consumers are likely to be unaware that the firm obtained the top rating in the past, at which point this firm's reputation is stuck at approximately the prior level for a long time.

1.3 Key features of the construction

My construction has a natural dynamic structure that speaks to a number of important economic phenomena and contributes to several strands of the literature.

Initial audit. First, this construction features a screening phase that precedes a certification phase. This screening phase sheds light on the initial auditing procedure that is common in many certification practices. In my model, for ratings to be able to induce sufficiently high reputations to sustain rewards and to induce sufficiently low reputations to sustain punishments, the rating system must have collected a sufficient amount of information concerning the firm's past trade outcomes. The certifier uses the screening phase to collect this information, ensuring that the ratings in the certification phase are effective in influencing consumers' beliefs.

Coarse ratings. My construction also illuminates the role of coarse ratings, namely ratings that censor the history of ratings and outcomes, in addressing dynamic moral hazard. By fully censoring this history and thus providing consumers with no information, a competent firm cannot build its reputation and therefore has no effort incentives. In contrast, by not censoring this history at all, new outcomes produced by the firm will eventually have negligible effect on its reputation given consumers' access to this history, disrupting this firm's effort incentives. The coarse ratings in my construction censor just the right amount of information in the history of ratings and outcomes to sustain the competent firm's effort incentives.

The prevalence of coarse ratings has long been appealing to economists. My analysis of dynamic moral hazard complements a number of insightful papers that show that coarse ratings can, for example, credibly convey information (Crawford and Sobel, 1982; Morgan and Stocken, 2003; Chakraborty and Harbaugh, 2007; Goel and Thakor, 2015), maximize certifiers' profits (Lizzeri, 1999), maximize firms' profits (Ekmekci, 2011), sustain relational contracts (Fong and Li, 2016), maximize participation in certification (Harbaugh and Rasmusen, 2018), address static moral hazard (Zapechelnyuk, 2020), address static adverse selection (Hopenhayn and Saeedi, forthcoming), and stimulate investments (Lorecchio and Monte, forthcoming).

Although my analysis is primarily motivated by markets for experience goods, it also sheds light on the role of rating coarseness in sustainability certifications and ecolabels to motivate firms' engagement in high environmental standards, or in corporate credit ratings to motivate firms to meet their debt obligations. In this latter regard, my construction offers a counter perspective on common criticisms from regulators that credit ratings are too coarse to be informative (see, e.g., Pagano and Volpin, 2010; Partnoy, 2017), highlighting the role of rating coarseness in giving firms incentives to avoid excessive risk-taking and to manage their investments diligently.

Graduated punishments. My construction also offers a novel rationale for graduated punishments, which are observed by many social scientists as common practice in successful long-run relationships (see, e.g., Ostrom, 1990, 2000; Ellickson, 1991; Agrawal, 2003). Dixit (2009), nonetheless, argues that this observation is difficult to reconcile with existing models of game theory; Abreu, Bernheim and Dixit (2005) anticipate that game-theoretic explanations of graduated punishments would arise in economic environments that feature both adverse selection and moral hazard. My model, indeed, features both adverse selection and dynamic moral hazard.

There is a small number of papers, mostly in the literature of law and economics, that motivate the use of graduated punishments with the premise that punishing offenders is socially costly, as first (informally) suggested by Stigler (1970). Rubinstein (1979) considers a setting in which punishing individuals who unintentionally commit offenses is socially costly, and so punishments are desirable only upon multiple offenses. Polinsky and Rubinfeld (1991), Chu, Hu and Huang (2000), Abreu et al. (2005), and van der Made (2019) consider various settings in which individuals have different types that capture their propensities to commit offenses, and these types are hidden from regulators. Because punishing offenders is costly, it is desirable to use the mildest punishment that is harsh enough to deter offenses. Repeated offenses reveal individuals' high types and thus the need for harsher punishments to deter offenses, leading to graduated punishments.

In contrast, in my model, punishments impose no intrinsic social cost; the competent firm is motivated to take the socially desirable action, namely full effort, even when it is punished with low reputations. Graduated punishments arise as a natural candidate to sustain effort incentives because Bayes' rule imposes a constraint on consumers' belief updating: as discussed, ratings must induce low reputations for effective punishments, but consumers' inferences limit the extent to which ratings can do so.

Low-standard honors certification. Finally, my construction can be interpreted as an honors certification scheme, which is also common practice. Safety and environmental organizations often accredit firms with a certified rating, and sometimes additionally award the top-performing ones. In my construction, the normal rating can be interpreted as "certified," and the top rating can be interpreted as "certified with honors." With this interpretation, my construction complements Harbaugh and Rasmusen (2018), who show that honors certification could help motivate sellers to participate in certification.

In my construction, the certified rating pools a competent firm with an inept firm to maintain a firm's reputation at approximately the prior level. In this sense, certification has a low standard, as the inept firm that is almost certain to fail the screening phase would occasionally be certified. This feature is, indeed, a common concern in various industries in practice; after all, a certified firm is often perceived as reliable for quality provision. Changing Market Foundation (2018), for instance, explicitly calls on certifiers in different markets to improve their standards. My construction highlights the role of this low standard to sustain effective punishments. Pooling as punishment has also been noted in other economic settings. For instance, Ghosh and Ray (1996) find that in community enforcement, cooperative behavior can be sustained by pooling opportunistic deviators with myopic, uncooperative individuals.

1.4 Related literature

The previous section has described the contribution of my main result to the literature. In this section, I relate my analysis to the literature from a modeling perspective.

Foremost, my analysis contributes to the literature on certification. Existing models concern certifying a firm's type in static adverse-selection settings (e.g., Lizzeri, 1999; Harbaugh and Rasmusen, 2018). Some models additionally feature an initial moral-hazard stage where a firm's type is determined (e.g., Albano and Lizzeri, 2001; Zapechelnyuk, 2020). These models abstract from the role of certification in providing firms with dynamic effort incentives, which is the focus of this paper: in my model, to motivate effort in any trade, future ratings must reflect the competent firm's present outcome to affect future consumers' information and payments. To focus on addressing dynamic moral hazard, my analysis abstracts from other dimensions in certification design, such as certification fees or legal requirements, and abstracts from other reasons to censor information, such as those mentioned in Section 1.3.

My analysis also contributes to the literature on reputations. Motivated by markets for experience goods, I model the interaction between the firm and the consumers as in Mailath and Samuelson (2001). Since Holmström (1982), the temporary nature of reputational incentives has prompted researchers to analyze mechanisms that sustain these incentives, such as competition (e.g., Hörner, 2002), limited or coarse memory (e.g., Liu, 2011; Monte, 2013; Liu and Skrzypacz, 2014; Pei, 2022a,b), changing player types (e.g., Cole, Dow and English, 1995; Holmström, 1982; Mailath and Samuelson, 2001; Phelan, 2006), and information design (e.g., Ekmekci, 2011; Hörner and Lambert, 2021). My construction contributes to this last strand of the literature, which I elaborate next.

Ekmekci (2011) studies rating design with different fundamentals and predictions. He studies a repeated product-choice game featuring a good firm type that commits to exert effort but no inept type.[2] His rating system helps a patient competent firm achieve approximately the Stackelberg payoff at each history of play but does not implement virtual full effort. To do so, almost all ratings in his rating system induce the firm to play the Stackelberg mixture between high and low efforts; this firm's incentives to mix are motivated by a high profit associated with a top rating that pools this firm with the good type and induces this firm to *shirk*. In turn, there is no need to provide intertemporal effort incentives at the top rating in his construction, and so his intermediate ratings do not serve as punishments upon a bad outcome at the top rating. In contrast, in my model, intertemporal effort incentives must be provided at the top rating, necessitating the use of other ratings as punishments. This necessity motivates my use of graduated punishments, given the constraints on belief updating imposed by consumers' inferences. Moreover, in my model, no rating system can help the firm achieve the Stackelberg payoff even in the limit of no discounting.

The crucial role of coarse ratings in my analysis contrasts with Hörner and Lambert

[2] More precisely, in his model, the good type commits to exert higher effort than the Stackelberg type does; the Stackelberg type mixes between high and low efforts.

(2021), who study rating design that maximizes a career-concerned agent's efforts over time. Among other modeling differences, they focus on linear ratings so that the rating in each time is a linear combination of the past signals about the agent's type, those about the agent's efforts, as well as those about the agent's outputs. Linear ratings, by definition, rule out coarse ratings.

Relatedly, Lorecchio and Monte (2022) study rating design in a bad-reputation setting à la Ely and Välimäki (2003). Incentives in their setting are fundamentally different from mine (and the above cited papers) in which ratings maintain "good" reputation effects to induce the firm to take the efficient action, i.e., full effort. Dellarocas (2005) studies rating design in a model in which, as in my model, the firm's effort is hidden but, different from my model, consumers have no uncertainty concerning the firm's type. This distinction is substantive. In his model, as well as in a special case of my model where the firm is known to be competent, no rating system can achieve efficiency even in the limit of no discounting. My analysis shows how ratings can leverage consumers' beliefs that the firm is possibly an inept type to virtually achieve efficiency.

Finally, albeit tangentially, my model relates to the literature that examines how bounded memory of trade outcomes affect incentives (e.g., Kovbasyuk and Spagnolo, 2018; Bhaskar and Thomas, 2019). Different from these models, consumers in my model do not observe past trade outcomes but observe past ratings, the information content of which can be designed by the certifier subject to the constraint on belief updating imposed by consumers' inferences. My focus on virtually implementing full effort also relates to a recent literature on dynamic games concerning the discounted frequency of the long-run player's actions (e.g., Li and Pei, 2021; Pei, 2022a,b).

2 Model and statement of main result

2.1 Trades

Time is discrete and the horizon is infinite. A long-lived firm (it) trades with a sequence of short-lived consumers (he), with a new consumer entering in every period. A certifier (she) chooses a rating system, described formally below in Definition 1. The firm has a private type θ, which is competent ($\theta = C$) with probability $\mu \in (0,1)$ and is inept ($\theta = I$) otherwise. Before trades happen, the certifier chooses a rating system.[3] This rating system induces a repeated game, consisting of periods $t \in \{0, 1, \dots\}$. In each period, the rating system updates the firm's rating before a new consumer enters. The consumer pays the firm his expected utility of the trade outcome upfront, as I specify below in (2). The firm then chooses effort, which generates a noisy outcome. Finally, the consumer leaves and the next period unfolds.

In each trade, there is a unit continuum of effort levels, $e \in [0,1]$. A competent firm chooses effort from the continuum; an inept firm only exerts zero effort. Higher effort is more likely to yield a good outcome. Specifically, effort e yields a good outcome $\bar{y}$ with probability $e(1-\rho) + (1-e)\rho$ and yields a bad outcome $\underline{y}$ otherwise, where $\rho \in (0, \frac{1}{2})$ is an error probability.[4] The consumer receives a utility of 1 from a good outcome and a utility of 0 from a bad outcome. The firm's profit equals the consumer's payment minus its effort cost. Effort e costs the firm ce, where $c > 0$. I refer to effort $e = 1$ as full effort, and I assume that full effort maximizes social surplus, namely the sum of the consumer's payoff and the firm's payoff, in the trade. That is, the social surplus associated with full effort exceeds that with zero effort:

$$1 - \rho - c > \rho. \tag{1}$$

In any induced game, the firm's type and efforts are hidden from the consumers. In each period, the firm observes the rating and the outcome; the consumer sees only a vector of ratings that I denote generically by $\vec{r}$, consisting of the current rating and the rating in each of the most recent past $K \geq 0$ periods. Thus, in each period, the consumer does not observe the outcome in the trade before leaving, but my results stay unchanged if he does.

[3] Because the rating system is chosen once and for all, the certifier effectively commits to the system. Commitment may arise from her reputational concerns (Coffee, 1997). A famous example of a certifier's desire to signal such commitment is the expulsion by Better Business Bureaus of its Los Angeles affiliate in 2009, after several eateries ware discovered to pay for high ratings. See "Better Business Bureau expels Los Angeles area chapter," *Los Angeles Times*, March 12, 2013.

[4] My results extend to settings with more than two outcomes, so long as each consumer's expected payoff when he expects effort 1 from the firm is $1 - \rho$ and that when he expects effort $e = 0$ from the firm is ρ. Moreover, the assumption that the error probability ρ is independent of effort e is for notational convenience.

I next define a rating system. Let R denote a set of ratings that is freely chosen by the certifier.

Definition 1. *A rating system is a collection $S = (S_t)_{t=0}^{\infty}$ where each*

$$S_t : R^t \times \{\bar{y}, \underline{y}\}^t \to \Delta(R)$$

maps the firm's past ratings and outcomes up to and including period $t-1$ to a distribution over a set of ratings R, from which the period-t rating is drawn.

The firm's rating in each period is determined by its past ratings and past outcomes. To be sure, even if $K = 0$ so that consumers observe only the current ratings, rating systems are defined to be flexible enough to disclose as much information as one might wish, namely all past ratings and past outcomes, via a current rating.

Let $G(S)$ denote the repeated game induced by a rating system S. I next describe strategies, payoffs and equilibrium in this game.

2.2 Induced game

In the game $G(S)$, in each period t, the consumer forms a belief that the firm is competent via Bayes' rule given his prior belief μ that the firm is competent, his rating observations $\vec{r}$, as well as his knowledge of the system S. This belief, denoted by $\varphi_t(\vec{r})$, is interpreted as the firm's reputation.

Let $H_t := (R \times [0,1] \times \{\bar{y}, \underline{y}\})^t \times R$ be the set of the firm's histories in period t before it chooses effort, consisting of past ratings, efforts and outcomes and the current rating. The firm's strategy is a collection of maps $\sigma = (\sigma_t)_{t=0}^{\infty}$, where each $\sigma_t : H_t \times \{C, I\} \to [0,1]$ specifies an effort in period t at each history $h_t \in H_t$ given its type θ, with the restriction that $\sigma_t(\cdot, I) = 0$. The strategy σ induces a probability measure P_θ over the set of infinite histories in the induced game, conditional on type θ. When there is no risk of ambiguity, I write the competent firm's strategy $\sigma_t(\cdot, C)$ simply as $\sigma_t(\cdot)$.

A consumer's payment, which is his expected utility from an outcome, depends on his belief on the firm's type and on the competent firm's effort. Specifically, a period-t

consumer, who sees a vector of ratings $\vec{r}$ and expects that the competent firm plays a strategy σ, pays the firm

$$p_t(\vec{r}) = \rho + (1-2\rho)\,\varphi_t(\vec{r})\,\mathbf{E}^{P_C}\left[\sigma_t(h_t)|\vec{r}\right], \tag{2}$$

where the expectation $\mathbf{E}^{P_C}$ is taken over the competent firm's period-t histories that determined its effort, with respect to the measure P_C induced by strategy σ. Given the utility normalization (utility of 1 from a good outcome and 0 from a bad outcome), this payment equals the consumer's belief of receiving a good outcome.

The firm has a discount factor $\delta \in (0,1)$. The competent firm chooses a strategy σ to maximize its average discounted sum of profits:

$$(1-\delta)\,\mathbf{E}^{P_C}\left[\sum_{t=0}^{\infty}\delta^t\left(p_t(\vec{r}_t) - ce_t\right)\right], \tag{3}$$

where e_t denotes the effort level chosen in period t and $\vec{r}_t$ denotes the vector of ratings observed by the period-t consumer.

Because the induced game has no observable deviations, I use Bayesian Nash equilibrium as the solution concept. A zero-effort equilibrium exists in which the competent firm always exerts zero effort.

2.3 The certifier's problem

I next describe the certifier's problem. In any equilibrium in the game induced by any rating system, the competent firm's average discounted sum of efforts is given by

$$(1-\delta)\mathbf{E}^{P_C}\left[\sum_{t=0}^{\infty}\delta^t e_t\right]. \tag{4}$$

Given any $\varepsilon > 0$, I say that a rating system S ε-implements full effort if in the induced game $G(S)$, an equilibrium exists in which the competent firm's average discounted sum of efforts (4) is at least $1-\varepsilon$.

The certifier's objective is to virtually implement full effort, described formally in Definition 2 below, by designing a rating system that ε-implements full effort for each $\varepsilon > 0$. Because a zero-effort equilibrium exists in any induced game, hereafter, I assume without loss of generality that $\varepsilon \in (0, 1)$.[5]

Definition 2. *Full effort is virtually implementable if for each $\varepsilon \in (0, 1)$, there exists a rating system S^ε such that in the induced game $G(S^\varepsilon)$, there exists $\underline{\delta}_\varepsilon \in (0, 1)$ such that for every $\delta \in [\underline{\delta}_\varepsilon, 1)$, the system S^ε ε-implements full effort.*

My focus on the firm being sufficiently patient is motivated by the fact that the competent firm's effort incentives must be provided intertemporally. In any equilibrium, in each trade, because this firm collects the consumer's payment upfront, it exerts positive effort if and only if its future payoff upon a good outcome sufficiently exceeds that upon a bad outcome and this firm is sufficiently concerned about its future payoff. Moreover, it takes time for a rating system to be able to induce sufficiently high reputations (and so high payoffs) for rewards and to induce sufficiently low reputations (and so low payoffs) for punishments: to do so, this rating system must have collected a sufficient amount of past outcomes so that it can display ratings that are sufficiently informative about the firm's type.

2.4 Statement of main result

My main result is:

Proposition 1. *Full effort is virtually implementable if and only if*

$$c < (1 - 2\rho)^2, \tag{MH}$$

$$c < \frac{(1 - \mu)(1 - 2\rho)^2}{1 - \rho - \mu(1 - 2\rho)}, \tag{RB}$$

$$K < \infty. \tag{CI}$$

These conditions elucidate the limit of ratings in providing incentives for consistent quality provision. The proof is in the Appendix. In Section 3 and Section 4, I sketch this proof. In particular, in showing sufficiency of these conditions for virtually

[5] Readers familiar with the robust mechanism design literature may wonder how the results would change if the certifier is concerned with the "worst" equilibrium that minimizes (4). This formulation is trivial in my setting, because a zero-effort equilibrium exists in any induced game.

implementing full effort, I construct, for each $\varepsilon \in (0,1)$, a rating system that ε-implements full effort.

3 Necessity

In this section, I discuss the necessity of (MH), (RB), and (CI) in Proposition 1. It is useful to consider for a moment a full-effort equilibrium in which the competent firm exerts full effort in *all* histories in Lemma 1 below.

Lemma 1. *In the game induced by any rating system, in a full-effort equilibrium, the competent firm's continuation payoff at each history is at most*

$$\bar{V} := 1 - \rho - c - \frac{\rho c}{1 - 2\rho}, \tag{5}$$

and is at least

$$\underline{V} := \underline{\pi} + \frac{(1-\rho)c}{1 - 2\rho}, \tag{6}$$

for some $\underline{\pi} > \rho - c$.

In a full-effort equilibrium, in each trade, the competent firm's payoff is at most $1 - \rho - c$, according to (2). This firm exerts full effort if and only if its continuation payoff upon a good outcome sufficiently exceeds that upon a bad outcome; the proof of Lemma 1 shows that their difference must be at least $c/[\delta(1-2\rho)]$. Thus, in every trade, this firm receives an expected, discounted continuation punishment of at least

$$\delta\rho\left(\frac{c}{\delta(1-2\rho)}\right) = \frac{\rho c}{1-2\rho}, \tag{7}$$

because it produces a bad outcome with probability ρ. The upper bound (5) thus follows. Similarly, in each trade, the competent firm's payoff strictly exceeds $\rho - c$. This is because, by (2), the consumer's payment must strictly exceed ρ, since this firm must have a positive reputation and the consumer correctly expects its full effort. For this firm to exert full effort in this trade, it must receive an expected, discounted continuation reward of at least

$$\delta(1-\rho)\left(\frac{c}{\delta(1-2\rho)}\right)=\frac{(1-\rho)c}{1-2\rho}, \tag{8}$$

because it produces a good outcome with probability $1-\rho$. The lower bound (6) thus follows.

Lemma 1 implies that absent adverse selection, i.e., if the firm is known to be competent, then it is impossible to virtually implement full effort. In this case, the firm's reputation is always one. If an equilibrium exists in which the competent firm exerts full effort in almost all trades, then by (2), this firm's payoff must be almost $1-\rho-c$, which is strictly larger than $\bar{V}$ and yields a contradiction. Intuitively, absent adverse selection, low payments in punishments require this firm to shirk, according to (2). Because the competent firm's full efforts could unluckily lead to bad outcomes, it must face nonnegligible punishments on path and so, contradictorily, nonnegligible periods of shirking. This observation complements the discussion in Section 1 concerning how my analysis contrasts with that of Dellarocas (2005).

I next turn to the necessity of the three conditions.

3.1 Dynamic moral hazard

Condition (MH) requires that effort cost is sufficiently small. This condition arises because the competent firm's dynamic moral hazard limits rewards and punishments that the certifier can use to motivate effort.

Consider, for each $\varepsilon \in (0,1)$, an equilibrium in which the competent firm's average discounted sum of efforts (4) exceeds $1-\varepsilon$; in this equilibrium, let $\bar{V}^{\varepsilon}$ denote this firm's highest continuation payoff across its histories and let $\underline{V}^{\varepsilon}$ denote its lowest continuation payoff across its histories. For each such ε, motivating the firm to incur an immediate positive effort cost in a trade requires that $\bar{V}^{\varepsilon} > \underline{V}^{\varepsilon}$, so that the firm can be rewarded upon a good outcome and punished upon a bad outcome. Virtual implementarity of full effort thus requires that

$$\bar{V}=\lim_{\varepsilon\downarrow 0}\bar{V}^{\varepsilon}\geq\lim_{\varepsilon\downarrow 0}\underline{V}^{\varepsilon}=\underline{V},$$

where $\bar{V}$ and $\underline{V}$ are given in (5) and (6); this inequality, alongside the fact that $\underline{\pi} > \rho - c$ as stated in Lemma 1, simplifies to (MH).

Given a higher error probability ρ, (MH) is more stringent: the competent firm has weaker effort incentives because the wedge between the trade payoff upper bound $1 - \rho - c$ and lower bound $\rho - c$ is smaller, and because outcomes are noisier signals of the firm's efforts.

3.2 Separating reputations

Condition (RB) also requires that effort cost is sufficiently small. It arises from the competent firm's desire to separate from the inept type for higher revenues.

To derive this condition, it is useful to consider again a full-effort equilibrium. Let V^θ denote the type-θ firm's initial payoff in this equilibrium. In this equilibrium, the firm's *ex ante* payoff is

$$
\begin{aligned}
(1-\mu)V^I + \mu V^C &= (1-\delta)\mathbf{E}\left[\sum_{t=0}^{\infty}\delta^t\left[p_t(\vec{r}_t) - \mu c\right]\right] \\
&= (1-\delta)\mathbf{E}\left[\sum_{t=0}^{\infty}\delta^t\left[\rho + (1-2\rho)\varphi_t(\vec{r}_t) - \mu c\right]\right] \\
&= \rho + \mu(1 - 2\rho - c). \qquad (9)
\end{aligned}
$$

The second line follows from (2), and the third line follows because the firm's average reputation must be equal to its prior reputation. Because the competent firm can always "mimic" an inept type by consistently exerting zero effort, $V^C \geq V^I$. Moreover, it must be true that $\bar{V} > V^C$, where $\bar{V}$ is given in (5). The reason is that for the competent firm's initial trade payoff to be equal to the upper bound $\bar{V}$ in equilibrium, the consumer's payment in the first period must be equal to $1 - \rho$. This latter event, however, requires this consumer to know that the firm is competent according to (2), which is impossible. As a result, $\bar{V} > V^C \geq V^I$, and by (9),

$$
\bar{V} > \rho + \mu(1 - 2\rho - c),
$$

requiring that the firm's payoff upper bound strictly exceeds its *ex ante* payoff. By using (5), this inequality simplifies to (RB). Finally, because $\lim_{\varepsilon\downarrow 0} \bar{V}^{\varepsilon} = \bar{V}$, virtual implementation of full effort requires (RB).

Given a higher error probability ρ, (RB) is more stringent: building a reputation becomes more difficult, as outcomes become noisier signals of efforts; moreover, reputation building becomes less useful in raising consumers' willingness to pay, as consumers expect to be more likely to receive bad outcomes despite the competent firm's full efforts. On the other hand, given a higher prior reputation μ, (RB) is also more stringent, as the competent firm's gain from building a reputation is smaller. In particular, if $\mu = 1$ so that consumers are certain that the firm is competent, then (RB) must fail and virtually implementing full effort is impossible, complementing the discussion preceding Section 3.1.

3.3 Consumers' information

Finally, condition (CI) requires that consumers observe only a finite number of recent past ratings. This condition arises because Bayes' rule limits the extent to which ratings can manipulate consumers' beliefs for rewards and punishments.

If consumers observe all past ratings, then new ratings will eventually have negligible effects on consumers' beliefs about the firm's type. More precisely, over time, consumers' beliefs about the firm's type given their rating observations must converge, because their beliefs are a bounded martingale and so converging in view of the martingale convergence theorem (see, e.g., Billingsley, 2008, Theorem 35.4). Thus, eventually, large enough variations in payments that sustain effective rewards and effective punishments in equilibrium require the competent firm's shirking, according to (2). This shirking, again, must constitute a nonnegligible proportion of time and so virtually implementing full effort is impossible.[6]

[6]This observation is reminiscent of the result from Cripps, Mailath and Samuelson (2004), who show in a canonical reputation model with imperfect monitoring that the posterior beliefs of short-lived players converge and reputation effects vanish eventually.

4 Sufficiency

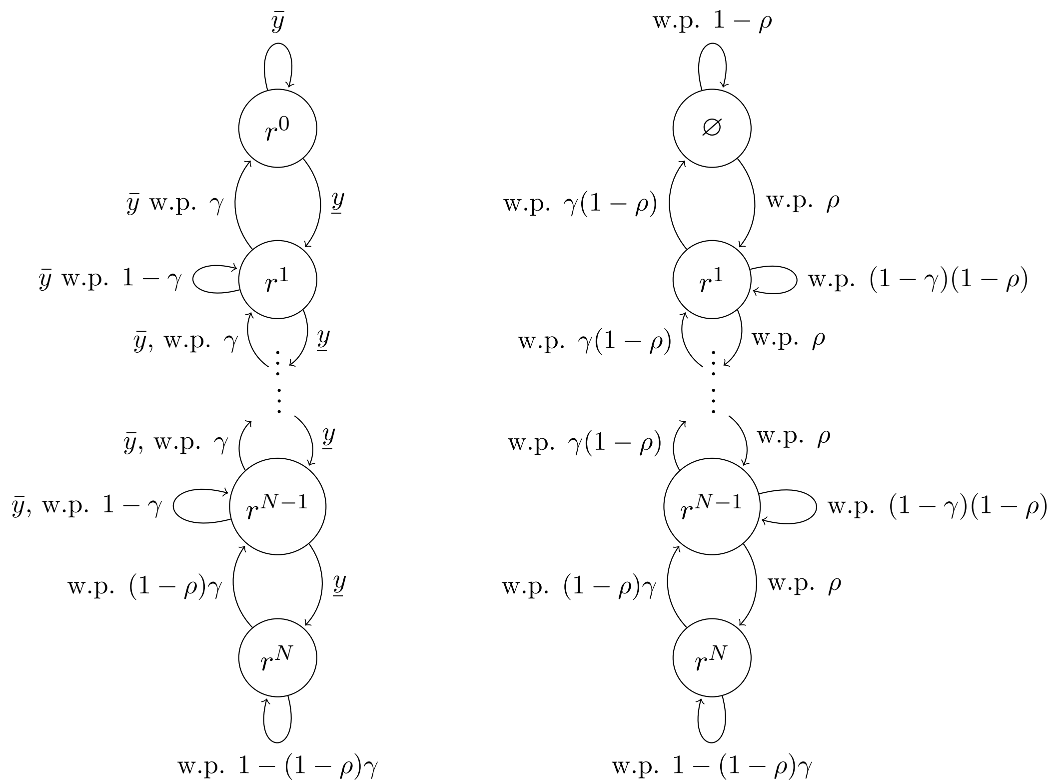
I next turn to the sufficiency of (MH), (RB), and (CI) in Proposition 1. Assume that these conditions hold, and fix $\varepsilon \in (0,1)$. In this section, I construct a rating system $\bar{S}^\varepsilon$ that ε-implements full effort.

4.1 Construction

Fix two finite integers L and N, chosen below in Lemma 3 to be sufficiently large. The rating system $\bar{S}^\varepsilon$ uses $\bar{R} = \{\varnothing, r^0, \dots, r^N\}$ as the set of ratings. Thus, this construction uses a large number of ratings; I will show in Section 5 that this construction can be implemented with only three ratings and in turn with an interpretation of an honors certification scheme, complementing the discussion in Section 1.3. The induced game $G(\bar{S}^\varepsilon)$ starts with a screening phase that consists of periods $t \in \{0, \dots, L-1\}$, followed by a certification phase that consists of periods $t \in \{L, L+1, \dots\}$. Also, fix two constants $\beta \in (\frac{1}{2}, 1)$ and $\gamma \in (\frac{\rho}{1-\rho}, 1]$, where β affects the qualification requirement in the screening phase and γ, specified in Appendix A, governs rating transitions in the certification phase.

In each period in the screening phase, the firm's rating is $\varnothing$. When this phase concludes, if the number of good outcomes in this phase is at least a threshold given by $(1-\rho)L - L^\beta$, then the firm is said to be "qualified"; otherwise, the firm is said to be "unqualified." Note that the firm knows whether it is qualified or is unqualified when the screening phase concludes, because it observes the outcomes that it produced.

In the certification phase, rating transitions follow a ladder structure, as depicted in Figure 1. A qualified firm's rating in the beginning of this phase, i.e., period L, is r^0. Given a current rating r^0, a good outcome leads to rating r^0 and a bad outcome leads to rating r^1 in the next period. Given a current rating r^n, $n = 1, \dots, N-1$, a good outcome leads to rating r^{n-1} with probability γ and to rating r^n with probability $1-\gamma$ in the next period, and a bad outcome leads to rating r^{n+1} in the next period. Finally, given a current rating r^N, regardless of the outcome, the next rating is r^{N-1} with probability $(1-\rho)\gamma$ and remains as r^N otherwise. On the other hand, an unqualified firm's rating in period L is $\varnothing$. Given a current rating $\varnothing$, the next rating is $\varnothing$ with



(a) Qualified firm **(b)** Unqualified firm

Figure 1: The rating system $\bar{S}^{\varepsilon}$

probability $1-\rho$ and is r^1 with probability ρ. Given a current rating r^1, the next rating is $\varnothing$ with probability $(1-\rho)\gamma$, r^1 with probability $(1-\rho)(1-\gamma)$ and r^{n+1} with probability ρ. Given a current rating r^n, $n = 2, \ldots, N-1$, the next rating is r^{n-1} with probability $(1-\rho)\gamma$, r^n with probability $(1-\rho)(1-\gamma)$ and r^{n+1} with probability ρ. Finally, given a current rating r^N, the next rating is r^{N-1} with probability $(1-\rho)\gamma$ and is r^N otherwise.

I next sketch a proof that this system $\bar{S}^{\varepsilon}$ ε-implements full effort. This proof proceeds via Lemmas 2 and 3 below. In these lemmas, I show that when the screening phase is long and the number of ratings is large, and if the firm is sufficiently patient, then there is an equilibrium that exhibits the following structure: the competent firm exerts full effort at "almost all" histories in the screening phase and exerts zero effort at other histories in that phase; in the certification phase, this firm exerts full effort if it is qualified and its current rating is not r^N; this firm exerts zero effort otherwise.

Note that an unqualified competent firm must consistently exert zero effort in the certification phase in any equilibrium in the induced game, because rating transitions upon not qualifying are independent of the trade outcomes and so its effort in each trade in that phase does not affect its future ratings and payoffs. Hereafter, I focus on the competent firm's effort incentives in the screening phase as well as in the certification phase upon qualification. Let $\bar{\sigma}^{\mathrm{CP}} = (\bar{\sigma}_t)_{t=L}^{\infty}$ denote the competent firm's strategies in the certification phase, as described above: this firm exerts full effort if it is qualified and its current rating is not r^N, and it exerts zero effort otherwise.

4.2 Graduated punishments

Consider first a qualified competent firm's incentives in the certification phase. It is useful to assume for a moment that consumers believe that the screening phase perfectly separates the two firm types: the competent firm is qualified with probability one and the inept firm is unqualified with probability one.

Lemma 2. *Suppose that consumers believe that the competent type is qualified with probability one and the inept type is unqualified with probability one, and that consumers conjecture that the competent firm plays according to $\bar{\sigma}^{\mathrm{CP}}$ in the certification phase. There exists N' such that if $N \geq N'$, then there exists $\delta'_N \in (0,1)$ such that for every $\delta \geq \delta'_N$, at each history in the certification phase upon qualification in the induced game $G(\bar{S}^{\varepsilon})$, the competent firm has no profitable deviation from $\bar{\sigma}^{\mathrm{CP}}$ and its incentive is strict.*

The certifier faces two main difficulties in motivating the qualified competent firm's full effort at all ratings except r^N. First, to sustain this firm's full effort at the top rating r^0, this firm must face sufficiently severe punishments with low reputations upon producing a bad outcome at this rating. However, consumers' Bayesian inferences require that on average, the competent firm's reputation must be at least its prior reputation μ, which can be quite high. In addition, virtually implementing full effort means that full effort needs to be motivated even in a trade in which this firm is already punished with a low reputation; in this trade, this firm must be further punished if it unluckily produces a bad outcome. Consumers' Bayesian inferences, again, limit how low this firm's reputation can fall in equilibrium.

My construction uses graduated punishments, described below, to overcome these difficulties. Rating transitions in the certification phase are chosen such that if consumers conjecture that the screening phase perfectly separates the two firm types and that the competent firm plays according to $\bar{\sigma}^{\mathrm{CP}}$ in the certification phase, then the following holds. In each trade during the certification phase, if the consumer's rating observations $\vec{r}$ include r^0, then he believes that the firm is competent for sure and pays the firm

$$\bar{p} := 1 - \rho \tag{10}$$

according to (2), because he expects full effort from this firm. If his rating observations $\vec{r}$ include neither rating r^0 nor rating $\varnothing$, then the firm's reputation is equal to the prior reputation μ and the consumer pays the firm

$$p_\mu := \rho + \mu(1 - 2\rho), \tag{11}$$

according to (2). Finally, if the current rating is r^N or if the consumer's rating observations $\vec{r}$ include rating $\varnothing$, then this consumer expects zero effort from this firm and pays the firm ρ. Note that $\bar{p} > p_\mu > \rho$. Thus, the top rating r^0 serves to reward the qualified competent firm and the other ratings serve to punish this firm. This firm's effort incentives arise from its desire to obtain the top rating r_0 and to avoid the other ratings.

Punishments are graduated in the following sense. At each intermediate rating $r^1, \ldots, r^{N-1}$ that punishes the firm, a bad outcome strengthens the punishment: given this bad outcome, this firm needs to produce at least one more good outcome to obtain the top rating r^0 in the ladder, as depicted in Figure 1a. Moreover, γ is chosen to small enough to make it difficult for the firm to exit punishments, but not too small to maintain effort incentives. This threat of further-lengthening punishments motivates this firm's full effort when it is currently punished with a low reputation induced by the intermediate ratings. Moreover, this threat ensures that punishments upon a bad outcome at the top rating are sufficiently severe, thereby motivating this firm's full effort at the top rating: because outcomes are noisy signals of efforts, upon a bad outcome at the top rating, after a certain number of trades, entering consumers are likely to be unaware that the firm obtained the top rating r^0 in the past, at which point this firm's reputation is stuck at approximately the prior reputation for a long time. This firm's effort incentives are sustained when it is sufficiently patient so that it does not discount the benefit from rating r^0 too much when its current rating is far down the ladder.

These graduated punishments cannot extend indefinitely, i.e., N must be finite. Otherwise, there are infinitely many ratings that are sufficiently far down the ladder in Figure 1a at which the competent firm becomes discouraged in obtaining the top rating r^0 and thus shirks. In my construction, at the bottom rating r^N, this firm knows that punishments would not be further strengthened upon a bad outcome. At this rating, if effort cost is sufficiently high, then this firm prefers to shirk. Accordingly, at this rating, the system uses exogenous rating transitions to enforce this firm's shirking.

4.3 Effort incentives

Because outcomes are noisy signals of efforts, screening only induces *imperfect* separation in any equilibrium: a competent firm is qualified with probability less than one and an inept firm is qualified with positive probability. Nonetheless, the perfect-separation benchmark discussed above is useful in establishing Lemma 3 below, which shows that $\bar{S}^{\varepsilon}$ ε-implements full effort.

Lemma 3. *There exist $\underline{L}_{\varepsilon}$ and $\underline{N}_{\varepsilon}$ such that for every $L \geq \underline{L}_{\varepsilon}$ and $N \geq \underline{N}_{\varepsilon}$, there exist $\underline{\delta}_{L,N,\varepsilon} \in (0,1)$ such that for every $\delta \geq \underline{\delta}_{L,N,\varepsilon}$, an equilibrium exists in which the competent firm plays according to $\bar{\sigma}^{\mathrm{CP}}$ in the certification phase in the induced game $G(\bar{S}^{\varepsilon})$. In this equilibrium, this firm's average discounted sum of efforts (4) is at least $1-\varepsilon$.*

Suppose the consumers conjecture that the competent firm exerts full effort in sufficiently many trades in the screening phase. Suppose also that the screening phase is long, and so the qualification threshold approximates $(1-\rho)L$, namely the expected number of good outcomes given consistent full effort from this firm in this phase. As a result, these consumers believe that the screening phase induces almost perfect separation: the competent firm is almost certain to be qualified and the inept firm is almost certain to be unqualified. Suppose these consumers also conjecture that the competent firm plays according to $\bar{\sigma}^{\text{CP}}$ in the certification phase. Then, in the certification phase, for any given rating observation, a consumer's payment is close to his payment in the perfect-separation benchmark in Section 4.2.[7] In turn, ratings upon qualifying induce higher profits for the competent firm than those upon not qualifying do. This benefit from qualification sustains the competent firm's best reply to exert full effort in sufficiently many trades in the screening phase, as conjectured by the consumers. In particular, because the qualification threshold approximates the expected number of good outcomes given the competent firm's consistent full effort in the screening phase, the competent firm exerts full effort in sufficiently many trades in this phase because it is unlikely to become discouraged in qualifying after unluckily producing many bad outcomes in that phase and then to abandon full effort.[8]

Turning to the certification phase, recall from Lemma 2 that a qualified, sufficiently patient competent firm's effort incentives are strict in the certification phase with a large number of ratings in the perfect-separation benchmark. Because consumers' payments in the certification phase with almost perfect separation approximate those in the perfect-separation benchmark, a qualified, sufficiently patient competent firm's best reply in the certification phase is to play according $\bar{\sigma}^{\text{CP}}$, as conjectured by the consumers.

As a result, there exists an equilibrium in which a sufficiently patient competent firm plays according to $\bar{\sigma}^{\text{CP}}$ in the certification phase in the induced game $G(\bar{S}^{\varepsilon})$.

[7]In fact, in each trade in the certification phase, if the current rating is r^N, the consumer's payment continues to be exactly ρ according to (2), as in the case when consumers believe that screening induces perfect separation.

[8]This threshold is inspired by Radner (1985), who uses this threshold in his construction of review strategies. Rating systems that build on his review strategies, nonetheless, cannot virtually implement full effort in general. These strategies typically divide time into review phases and display a rating at the end of each phase to reflect the outcomes in that phase. A short such phase contains limited information and thus disrupts effort incentives unless effort cost is low; a longer phase also leads to shirking, because a firm that produced several bad outcomes expects to fail the review and shirks for the rest of the phase.

In this equilibrium, when the discount factor is large, the competent firm's average discounted sum of efforts (4) is at least $1-\varepsilon$. The reasons are as follows. First, when the discount factor is large, the contribution of this firm's efforts during the screening phase to (4) is negligible. Moreover, as discussed above, the competent firm is almost certain to qualify given a long screening phase. Thus, to verify that the system $\bar{S}^{\varepsilon}$ ε-implements full effort, it suffices to verify that a qualified competent firm almost never receives rating r^N in the certification phase, given which this firm shirks in the equilibrium. This latter event holds when the number of ratings N is large, because a qualified competent firm's full efforts at ratings $r^0, \dots, r^{N-1}$ tend to drift its rating towards the top rating r^0 in the ladder depicted in Figure 1a.

5 Discussion

In this final section, I discuss further several notable features of my construction, complementing the discussion in Sections 1.3 and 1.4.

5.1 A three-ratings implementation

I briefly sketch that for each $\varepsilon > 0$, the above construction can be implemented with only three ratings. Suppose that the certifier privately uses the system $\bar{S}^{\varepsilon}$ in Section 4 and displays a rating (to the firm and the consumers) via another system S_3^{ε} as follows. As in the above construction, the induced game $G(S_3^{\varepsilon})$ starts with a screening phase that consists of periods $t \in \{0, \dots, L-1\}$, followed by a certification phase that consists of periods $t \in \{L, L+1, \dots\}$. In any trade during the screening phase, the system S_3^{ε} displays only a rating "bottom." In any trade during the certification phase, if $\bar{S}^{\varepsilon}$ delivers rating r^0, then S_3^{ε} displays a rating "top." if $\bar{S}^{\varepsilon}$ delivers rating r^n, $n = 1, \dots, N-1$, then S_3^{ε} displays a rating "normal." Finally, if $\bar{S}^{\varepsilon}$ delivers rating r^N or $\varnothing$, then S_3^{ε} displays rating "bottom." In the certification phase, the competent firm exerts full effort if it is qualified and its current rating is not the bottom rating; this firm exerts zero effort otherwise. In the screening phase, the competent firm plays as in the equilibrium identified in Lemma 3, exerting full effort in sufficiently many trades in that phase and yielding almost perfect separation.

Given almost perfect separation, payments in the certification phase are as follows. If a consumer observes a top rating and the current rating is not the bottom rating, then his payment is approximately $\bar{p}$ given in (10). If this consumer does not observe the top rating and the current rating is not the bottom rating, then his payment is approximately p_μ given in (11), as he forms expectations over ratings $r^1, \ldots, r^{N-1}$ that the system $\bar{S}^\varepsilon$ may deliver, and each of these ratings induces a payment that approximates p_μ. Finally, if this consumer observes that the current rating is the bottom rating, then he pays the firm ρ.

A qualified, sufficiently patient competent firm's effort incentives in the certification phase are preserved under this new system S_3^ε. Given a current top rating, the competent firm infers that its current rating under $\bar{S}^\varepsilon$ is r^0 and thus optimally exerts full effort. Given a current normal rating, this firm forms an expectation over its ratings under $\bar{S}^\varepsilon$ and faces an "average" incentive constraint for full effort. Because this firm's incentive constraints for full effort are satisfied under $\bar{S}^\varepsilon$, this average constraint holds. Given a current bottom rating, this firm knows that its rating under $\bar{S}^\varepsilon$ is r^N and thus optimally shirks. On the other hand, an unqualified competent firm must consistently exert zero effort in the certification phase, for the same reason as in Section 4. Finally, a sufficiently patient competent firm's incentives to exert full effort in most trades in a sufficiently long screening phase are preserved under this new system S_3^ε. The reason is that the competent firm benefits from qualification, as in Section 4: by the law of iterated expectations, the competent firm's expected continuation payoff upon qualifying under the new system S_3^ε is equal to that under the system $\bar{S}^\varepsilon$, and this firm's expected continuation payoff upon not qualifying under S_3^ε is equal to its counterpart under $\bar{S}^\varepsilon$.

5.2 Stackelberg payoff versus efficiency

A key theme in the reputations literature concerns whether players achieve their Stackelberg payoffs. The focus of my analysis, in contrast, concerns achieving efficiency.

Lemma 1 shows that in my model, the competent firm's equilibrium payoff is strictly smaller than the Stackelberg payoff in any induced game irrespective of discounting. In each trade, if this firm can publicly commit to some effort, then it would commit to full effort. This firm's Stackelberg payoff is thus $1 - \rho - c$, exceeding its equilibrium continuation payoff upper bound (5). Intuitively, attaining the Stackelberg payoff requires the competent firm's full effort in all trades as well as ratings that perfectly reveal its type in all these trades. As discussed in Section 3, these two requirements are mutually exclusive. This observation complements the discussion in Section 1 concerning the comparison of my analysis with Ekmekci (2011), who construct ratings that help a patient competent firm achieve approximately the Stackelberg payoff at each history of play but do not implement virtual full effort.

5.3 More firm types

Finally, I note that my construction in Section 4 remains effective if there are more than two firm types. To see this, suppose that there is a new firm type that can also choose effort from a unit continuum, and whose marginal effort cost c' differs from the competent firm's marginal effort cost c but satisfies (MH), (RB), and (CI). In addition, suppose that the consumers' prior belief on the firm's type is such that the firm is a competent type with some probability $\mu_C \in (0, 1)$, a new type with some probability $\mu_N \in (0, 1)$, and an inept type with probability $1 - \mu_C - \mu_N$. Without loss of generality, suppose that $c' < c$ and that $\mu_C + \mu_N = \mu$. The certifier can simply use the construction in Section 4 and rates this new type as if it is a competent type, thereby perfectly pooling the new type and the competent type in all trades. Because the new type has a smaller effort cost, if the construction motivates full effort from the competent type in virtually all trades, it also motivates full effort from the new type in all these trades.

Appendices

A Omitted details: the construction

Assume that (MH) and (RB) hold. Fix some $\hat{c} \in (c, \min[1, \frac{1-\mu}{1-\mu-\rho+2\mu\rho}](1-2\rho)^2)$. Define

$$\gamma := \min\left[1, \frac{(1-\mu)(1-2\rho)^2(1+K\rho)}{\left(\hat{c}+K(1-\mu)(1-2\rho)^2\right)(1-\rho)}\right]. \tag{12}$$

By construction, $\gamma(1-\rho) > \rho$.

Note that in the certification phase, the rating transitions induced by a qualified, competent firm are identical to those induced by an unqualified, inept firm, except that rating r^0 is replaced by rating $\varnothing$, conditional on consumers believing that the competent firm plays according to $\bar{\sigma}^{\mathrm{CP}}$ in that phase. Thus, if consumers believe that a competent type is qualified from the screening phase with probability one and an inept type is unqualified with probability one, and that the competent firm plays according to $\bar{\sigma}^{\mathrm{CP}}$ in the certification phase, then for any rating observations $\vec{r}_t$ of a period-t consumer that include neither r^0 nor $\varnothing$ in the certification phase, $P_C(\vec{r}_t) = P_I(\vec{r}_t)$, where the probability measures P_C and P_I are induced by the consumers' belief about both types' qualification probabilities and their conjecture that the firm's strategy in the certification phase. In turn, the firm's reputation is

$$\varphi_t(\vec{r}_t) = \frac{\mu P_C(\vec{r}_t)}{\mu P_C(\vec{r}_t) + (1-\mu)P_I(\vec{r}_t)} = \mu.$$

B Proofs

B.1 Proof of Lemma 1

Consider an equilibrium in which the competent firm exerts full effort at all histories. By exerting full effort following a history h in period t in which the consumer's rating observations are $\vec{r}$, the competent firm's continuation payoff is

$$(1-\delta)(p_t(\vec{r})-c)+\delta\left[(1-\rho)V(h\bar{y})+\rho V(h\underline{y})\right], \tag{13}$$

where $V(hy)$ denotes this firm's continuation payoff upon producing outcome y after history h. A one-stage deviation to exert zero effort yields a continuation payoff

$$(1-\delta)p_t(\vec{r})+\delta\left[\rho V(h\bar{y})+(1-\rho)V(h\underline{y})\right].$$

The firm's incentive constraint to exert full effort at history h is

$$V(h\bar{y})-V(h\underline{y})\geq\frac{(1-\delta)c}{\delta(1-2\rho)}. \tag{14}$$

By applying (14) to (13) recursively, and because this firm's payoff in each trade is at most $1-\rho-c$ according to (1),

$$\begin{aligned}&(1-\delta)(p_t(\vec{r})-c)+\delta\left[(1-\rho)V(h\bar{y})+\rho V(h\underline{y})\right]\\ \leq\ &(1-\delta)(1-\rho-c)+\delta\left[V(h\bar{y})+\frac{(1-\delta)\rho c}{\delta(1-2\rho)}\right]\leq\cdots\leq 1-\rho-c+\frac{\rho c}{1-2\rho}.\end{aligned}$$

This proves (5). Similarly, (13) is strictly larger than

$$\begin{aligned}&(1-\delta)(\rho-c)+\delta\left[(1-\rho)V(h\bar{y})+\rho V(h\underline{y})\right]\\ \geq\ &(1-\delta)(\rho-c)+\delta\left[V(h\underline{y})+\frac{(1-\delta)(1-\rho)c}{\delta(1-2\rho)}\right]>\cdots\geq\rho-c+\frac{(1-\rho)c}{1-2\rho}.\end{aligned}$$

These inequalities follow by applying (14) recursively and also because, according to (2), the consumers' payment in each trade strictly exceeds ρ given that the competent firm consistently exerts full effort. This proves (6).

B.2 Proof of Proposition 1

Claim 1 below is essential.

Claim 1. *Consider an equilibrium in which the competent firm exerts positive effort* *at some history $h_{\hat{t}}$ in some period $\hat{t}$. In this equilibrium, there must exist $t > \hat{t}$ such that this firm exerts positive effort at some history h_t in period t.*

Proof of Claim 1. Suppose, towards a contradiction, that there is an equilibrium in which the competent firm exerts positive effort at some history $h_{\hat{t}}$ in some period $\hat{t}$, but in all periods $t \geq \hat{t}$ following history $h_{\hat{t}}$, the competent firm exerts no effort. Because consumers' expectations about the firm's strategy are correct in equilibrium, their payments are equal to ρ in all period $t \geq \hat{t}$ according to (2). Thus, at history $h_{\hat{t}}$, with associated consumer's rating observations $\vec{r}_{\hat{t}}$, the competent firm's continuation payoff is $(1-\delta)(p_{\hat{t}}(\vec{r}_{\hat{t}}) - ce_{\hat{t}}) + \delta\rho$. A deviation to exert zero effort at this history is profitable, as this deviation gives this firm a continuation payoff of $(1-\delta)p_{\hat{t}}(\vec{r}_{\hat{t}}) + \delta\rho$, yielding a contradiction. ∎

B.2.1 Necessity

The necessity of (MH) and (RB) are clear from the main text. Here, I show that (CI) is necessary. Suppose, towards a contradiction, that $K = \infty$ but for every $\varepsilon > 0$, there exists a rating system given which, in the induced game, there is $\underline{\delta}$ such that for every $\delta \geq \underline{\delta}$, there is an equilibrium $\bar{\sigma}^{\varepsilon}$ in which the competent firm's average discounted effort (4) is at least $1-\varepsilon$. For each $\varepsilon > 0$, let $\bar{P}^{\varepsilon}_{C}$ denote the probability distribution over the infinite histories of the induced game conditional on a competent type in the equilibrium $\bar{\sigma}^{\varepsilon}$. Fix some $\kappa \in (0, \frac{1-\delta}{\delta}c)$ and some $\zeta \in (0, \frac{1-\delta}{\delta}c - \kappa)$. Consider a sufficiently small ε given which, for any two competent firm's histories h and h',

$$(1-\delta)\sum_{t=T+1}^{\infty}\delta^{t-(T+1)}\left\{\mathbf{E}^{\bar{P}^{\varepsilon}_{C}}\left[e_t|h\right] - \mathbf{E}^{\bar{P}^{\varepsilon}_{C}}\left[e_t|h'\right]\right\} < \frac{\zeta}{c(1-2\rho)} \tag{15}$$

in the associated equilibrium $\bar{\sigma}^{\varepsilon}$.

Let $V^{\varepsilon}(h_T(e_T, y_T))$ denote the competent firm's continuation payoff after exerting effort e_T and producing outcome y_T at history h_T, where T is chosen according to Claim 2 below to be sufficiently large. Because effort cost is linear, it is without loss to assume that the firm exerts full effort at any history given which it exerts positive effort. This firm's continuation payoff from exerting full effort at history h_T is

$$(1-\delta)\left(p(\vec{r}_T)-c\right)+\delta(1-\rho)V^{\varepsilon}(h_T(1,\bar{y}))+\delta\rho V^{\varepsilon}(h_T(1,\underline{y})).$$

By deviating to choose zero effort, its continuation payoff is

$$(1-\delta)p(\vec{r}_T)+\delta\rho V^{\varepsilon}(h_T(0,\bar{y}))+\delta(1-\rho)V^{\varepsilon}(h_T(0,\underline{y})),$$

This firm's incentive constraint for full effort at history h_T is therefore

$$\begin{aligned}(1-\rho)[V^{\varepsilon}(h_T(1,\bar{y}))-V^{\varepsilon}(h_T(0,\underline{y}))]&\\ +\rho[V^{\varepsilon}(h_T(1,\underline{y}))-V^{\varepsilon}(h_T(0,\bar{y}))]&\geq\frac{1-\delta}{\delta}c.\end{aligned}\tag{16}$$

Claim 2 below completes the proof that (CI) is necessary by establishing that the left side of (16) is strictly smaller than

$$(1-\rho)(\kappa+\zeta)+\rho(\kappa+\zeta)=\kappa+\zeta<\frac{1-\delta}{\delta}c,$$

yielding a contradiction to (16), as desired.

Claim 2. *There exists a sufficiently large period T such that for any $(e_T,y_T),(\tilde{e}_T,\tilde{y}_T)$,*

$$V^{\varepsilon}(h_T(e_T,y_T))-V^{\varepsilon}(h_T(\tilde{e}_T,\tilde{y}_T))<\kappa+\zeta.\tag{17}$$

Proof of Claim 2. Conditional on the firm being competent, consumers' beliefs $(\varphi_t^{\varepsilon}(\vec{r}_t))_{t=0}^{\infty}$ of the firm being competent are a bounded martingale and so converge to some $\mu_{\infty}^{\varepsilon}\in[0,1]$ almost surely by the martingale convergence theorem (see, e.g., Billingsley, 2008, Theorem 35.4). Because these beliefs are bounded above by one, the dominated convergence theorem (see, e.g., Billingsley, 2008, Theorem 16.4) ensures that these beliefs also converge in mean. By Claim 1, there exists a history h_T at some sufficiently large period T, the competent firm exerts positive effort, and for every $t \geq T$, for any concatenation $\hat{h}_t$ of h_T,

$$\mathbf{E}^{\bar{P}^\varepsilon_C}\left[|\varphi^\varepsilon_t(\vec{r}_t) - \mu^\varepsilon_\infty|\Big|\hat{h}_t\right] < \frac{\kappa}{1-2\rho}. \tag{18}$$

Then, letting $\omega_T := (e_T, y_T)$ and $\tilde{\omega}_T := (\tilde{e}_T, \tilde{y}_T)$,

$$\begin{aligned}
&V(h_T\omega_T) - V(h_T\tilde{\omega}_T)\\
&= (1-\delta)\mathbf{E}^{\bar{P}^\varepsilon_C}\left[\sum_{t=T+1}^{\infty}\delta^{t-(T+1)}\left(p_t(\vec{r}_t) - ce_t\right)\Bigg|h_T\omega_T\right]\\
&\qquad\qquad - (1-\delta)\mathbf{E}^{\bar{P}^\varepsilon_C}\left[\sum_{t=T+1}^{\infty}\delta^{t-(T+1)}\left(p_t(\vec{r}_t) - ce_t\right)\Bigg|h_T\tilde{\omega}_T\right]\\
&= (1-2\rho)(1-\delta)\sum_{t=T+1}^{\infty}\delta^{t-(T+1)}\Bigg\{\left(\mathbf{E}^{\bar{P}^\varepsilon_C}\left[\varphi_t(\vec{r}_t)|h_T\omega_T\right] - \mathbf{E}^{\bar{P}^\varepsilon_C}\left[\varphi_t(\vec{r}_t)|h_T\tilde{\omega}_T\right]\right)\\
&\qquad\qquad\qquad - c\left(\mathbf{E}^{\bar{P}^\varepsilon_C}\left[e_t|h_T\omega_T\right] - \mathbf{E}^{\bar{P}^\varepsilon_C}\left[e_t|h_T\tilde{\omega}_T\right]\right)\Bigg\}\\
&\leq (1-2\rho)\Bigg[\frac{\zeta}{1-2\rho} + (1-\delta)\times\\
&\qquad \sum_{t=T+1}^{\infty}\delta^{t-(T+1)}\left\{\mathbf{E}^{\bar{P}^\varepsilon_C}\left[|\varphi_t(\vec{r}_t) - \mu_\infty||h_T\omega_T\right] + \mathbf{E}^{\bar{P}^\varepsilon_C}\left[|\varphi_t(\vec{r}_t) - \mu_\infty||h_T\tilde{\omega}_T\right]\right\}\Bigg]\\
&< (1-2\rho)\left[\frac{\kappa}{1-2\rho} + \frac{\zeta}{1-2\rho}\right]\\
&= \kappa + \zeta,
\end{aligned}$$

as desired, where the third line follows from (2), the fourth line follows from (15) and the triangular inequality, and the fifth line follows from (18). ■

B.2.2 Sufficiency

Proof of Lemma 2. Fix some large $N > K+1$ that is chosen below. The state variables that govern a competent firm's continuation payoff in the induced game are given by a vector (k, r), consisting of the current rating r and the number $k \geq 1$ of past periods since rating r^0 is most recently realized, as these two numbers pin down consumers' beliefs and the firm's effort. The state transitions are as follows. For any

k, given a current state (k, r^0), the next state is $(1, r^0)$ upon a good outcome and is $(1, r^1)$ upon a bad outcome. Given a current state (k, r^n), $n = 1, \ldots, N-1$, upon a good outcome, the next state is $(k+1, r^{n-1})$ with probability γ and is $(k+1, r^n)$ otherwise; upon a bad outcome, the next state is $(k+1, r^{n+1})$. Finally, given a current state (k, r^N) for any k, the next state is $(k+1, r^{N-1})$ with probability $\gamma(1-\rho)$ and is $(k+1, r^N)$ otherwise.

Let $V_{(k,n)}$ denote the continuation payoff of the competent firm at a state (k, r^n) in the induced game. The continuation payoffs satisfy $V_{(k,0)} =: V_0$ for each $k = 1, \ldots, K$, and $V_{(k,n)} =: V_n$ for each $k \geq K+1, n = 1, \ldots, N$, as well as the system of Bellman equations

$$\begin{aligned}
V_0 &= (1-\delta)(1-\rho-c) + \delta\left((1-\rho)V_0 + \rho V_{(1,1)}\right), \\
V_{(k,n)} &= (1-\delta)(1-\rho-c) \\
&\quad + \delta\left((1-\rho)\gamma V_{(k+1,n-1)} + (1-\rho)(1-\gamma)V_{(k+1,n)} + \rho V_{(k+1,n+1)}\right), \\
&\qquad\qquad\qquad\qquad\qquad\qquad k = 1, \ldots, K, n = 1, \ldots, k, \\
V_n &= (1-\delta)(p_\mu - c) + \delta\left((1-\rho)\gamma V_{n-1} + (1-\rho)(1-\gamma)V_n + \rho V_{n+1}\right), \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad n = 1, \ldots, N-1, \\
V_N &= (1-\delta)\rho + \left((1-\rho)\gamma V_{N-1} + (1-(1-\rho)\gamma)V_N\right).
\end{aligned}$$

This linear system of $\frac{K(K+1)}{2} + N + 1$ equations has $\frac{K(K+1)}{2} + N + 1$ unknowns, namely, the continuation payoffs, and admits a unique solution.

To complete the proof of this claim, I verify that the incentive constraints for full effort in the limit of no discounting hold strictly, when N is (fixed and) sufficiently large:

$$\lim_{\delta\to 1} \delta(1-2\rho)\left(\frac{V_0 - V_{(1,1)}}{1-\delta}\right) > c, \tag{19}$$

$$\lim_{\delta\to 1} \delta(1-2\rho)\left(\frac{\gamma V_{(k+1,n-1)} + (1-\gamma)V_{(k+1,n)} - V_{(k+1,n+1)}}{1-\delta}\right) > c, \tag{20}$$

$$k = 1,\dots,K, n = 1,\dots,k,$$

$$\lim_{\delta\to 1} \delta(1-2\rho)\left(\frac{\gamma V_{n-1} + (1-\gamma)V_n - V_{n+1}}{1-\delta}\right) > c, \tag{21}$$

$$n = 1,\dots,N-1.$$

Since the competent firm's strategy induces an irreducible Markov chain over a finite set of $K(K+1)/2 + N + 1$ states upon a competent report, Ross (2014, Theorem 2.4) shows that the absolute unnormalized difference of continuation profits $\left|\frac{V_{(k,n)}-V_0}{1-\delta}\right|$ is uniformly bounded over k, n and $\delta \in (0,1)$. In turn, by Ross (2014, Theorem 2.2), the following holds. First, there exist bounded $\ell^N_{(k,n)}$ and g^N such that

$$\ell^N_{(k,n)} = \lim_{\delta\to 1} \frac{V_{(k,n)} - V_0}{1-\delta}, \quad \text{and} \quad \ell^N_{(k,n)} := \ell^N_n, \quad \forall k \geq K+1, n = 1,\dots,N.$$

Second, the following system of $\frac{K(K+1)}{2} + N + 1$ linear equations

$$\begin{aligned}
g^N &= 1 - \rho - c + \rho\ell^N_{(1,1)}, \\
g^N + \ell^N_{(k,n)} &= 1 - \rho - c + (1-\rho)\gamma\ell^N_{(k+1,n-1)} \\
&\quad + (1-\rho)(1-\gamma)\ell^N_{(k+1,n)} + \rho\ell^N_{(k+1,n+1)}, k = 1,\dots,K, n = 1,\dots,k. \\
g^N + \ell^N_n &= p_\mu - c + (1-\rho)\gamma\ell^N_{n-1} + (1-\rho)(1-\gamma)\ell^N_n + \rho\ell^N_{n+1}, \quad n = 1,\dots,N-1, \\
g^N + \ell^N_N &= \rho + (1-\rho)\gamma\ell^N_{N-1} + (1-(1-\rho)\gamma)\ell^N_N,
\end{aligned}$$

holds and admits a unique solution $(g^N, ((\ell^N_{(n,k)})^k_{n=1})^K_{k=1}, (\ell_n)^N_{n=1})$. Verifying (19)—(21)

is then equivalent to verifying

$$-\ell^N_{(1,1)} > \frac{c}{1-2\rho}, \tag{22}$$

$$\gamma\ell^N_{(k,n-1)} + (1-\gamma)\ell^N_{(k,n)} - \ell^N_{(k,n+1)} > \frac{c}{1-2\rho} \quad k = 1,\dots,K, n = 1,\dots,k, \tag{23}$$

$$\gamma\ell^N_{n-1} + (1-\gamma)\ell^N_{n} - \ell^N_{n+1} > \frac{c}{1-2\rho}, \quad n = 1,\dots,N-1. \tag{24}$$

At the solution of the above system, it holds that

$$\begin{aligned}
&\lim_{N\to\infty} -\ell^N_{(1,1)} = \frac{(1-\mu)(1-2\rho)(\gamma K(\rho-1)+K\rho+1)}{\gamma(1-\rho)} \\
&= \begin{cases} \frac{(1-\mu)(1-2\rho)(1-K(1-2\rho))}{1-\rho}, & \text{if } \hat{c} < \frac{(1-\mu)(1-2\rho)^2(1-K(1-2\rho))}{1-\rho}, \\ \frac{\hat{c}}{1-2\rho}, & \text{otherwise.} \end{cases} \\
&> \frac{c}{1-2\rho}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\lim_{N\to\infty} \left[\gamma\ell^N_{n-1} + (1-\gamma)\ell^N_{n} - \ell^N_{n+1}\right] \\
&= \lim_{N\to\infty} \left[\gamma\ell^N_{(k,n-1)} + (1-\gamma)\ell^N_{(k,n)} - \ell^N_{(k,n+1)}\right], \quad k = 1,\dots,K, n = 1,\dots,k, \\
&= \frac{1+\gamma}{\gamma}\left(\frac{(1-\mu)(1-2\rho)(1+K\rho)}{1-\rho}\right) \\
&> \frac{(1-\mu)(1-2\rho)(\gamma K(\rho-1)+K\rho+1)}{\gamma(1-\rho)} = \lim_{N\to\infty} -\ell^N_{(1,1)} > \frac{c}{1-2\rho}.
\end{aligned}$$

Because these constraints hold strictly in the limit as $N \to \infty$, there is $N > K+1$ sufficiently large such that (22)—(24) hold. ■

Proof of Lemma 3. To prove Lemma 3, I first prove Claims 3 and 4 below, which are essential.

Claim 3. *Consider an equilibrium in which the competent firm plays according to $\bar{\sigma}^{\mathrm{CP}}$ in the certification phase. In this equilibrium, let q_θ denote the probability that a type-θ firm is qualified when the screening phase concludes, and let $p_t(\vec{r}_t|q_C, q_I)$ denote* *the consumer's payment in period t in the certification phase upon observing ratings $\vec{r}_t$ given qualification probabilities (q_C, q_I). For every $\eta > 0$, there exist $\underline{q}_{C,\eta} \in (0,1)$ and $\bar{q}_{I,\eta} \in (0,1)$ such that if $q_C \geq \bar{q}_{C,\eta}$ and $q_I \leq \underline{q}_{I,\eta}$, then in each period t in the certification phase, for any $\vec{r}_t$, $|p_t(\vec{r}_t|q_C, q_I) - p_t(\vec{r}_t|1,0)| < \eta$.*

Proof of Claim 3. Fix $\eta > 0$ and fix an equilibrium as stated in the claim. Let $P^\theta[\vec{r}_t|Q]$ denote the probability that a consumer observes ratings $\vec{r}_t$ in period t during the certification phase given that the firm's type is θ and this firm is qualified when the screening phase concludes; let $P^\theta[\vec{r}_t|U]$ denote the counterpart given that this firm is unqualified when the screening phase concludes. By Bayes' rule, for each period t in the certification phase,

$$\begin{aligned}
&\varphi_t(\vec{r}_t|q_C, q_I) \\
= &\frac{\mu P^C[\vec{r}_t]}{\mu P^C[\vec{r}_t] + (1-\mu)P^I[\vec{r}_t]} \\
= &\frac{\mu(P^C[\vec{r}_t|Q]q_C + P^C[\vec{r}_t|U](1-q_C))}{\mu(P^C[\vec{r}_t|Q]q_C + P^C[\vec{r}_t|U](1-q_C)) + (1-\mu)(P^I[\vec{r}_t|Q]q_I + P^I[\vec{r}_t|U](1-q_I))}.
\end{aligned}$$

Note that $P^\theta[\vec{r}_t|Q]$ and $P^\theta[\vec{r}_t|U]$ are independent of q_C and q_I, because the rating transitions in the certification phase are constructed to be independent of (q_C, q_I). Thus, there exist $\bar{q}_{C,\eta}, \underline{q}_{I,\eta} \in (0,1)$ such that for every $q_C \geq \bar{q}_{C,\eta}$ and $q_I \leq \underline{q}_{I,\eta}$,

$$|\varphi_t(\vec{r}_t|q_C, q_I) - \varphi_t(\vec{r}_t|1,0)| < \frac{\eta}{1-2\rho}$$

for every t. Finally, because

$$p_t(\vec{r}_t|q_C, q_I) = \begin{cases} \rho + (1-2\rho)\varphi_t(\vec{r}_t|q_C, q_I), & \text{if } r_t \neq r^N, \\ \rho, & \text{otherwise,} \end{cases}$$

it follows that if $q_C \geq \bar{q}_{C,\eta}$ and $q_I \leq \underline{q}_{I,\eta}$, $|p_t(\vec{r}_t|q_C, q_I) - p_t(\vec{r}_t|1,0)| < \eta$ for every t, as desired. ■

Claim 4. *In any equilibrium in which the competent firm plays according to $\bar{\sigma}^{\text{CP}}$ in the certification phase, for every $\xi \in (0,1)$, there exist $\hat{L}_\xi$ and $\hat{N}_\xi$ such that for every*

$L \geq \hat{L}_\xi$ and every $N \geq \hat{N}_\xi$, there is $\hat{\delta}_{L,N,\xi} \in (0,1)$ such that for every $\delta \geq \hat{\delta}_{L,N,\xi}$, $q_C > 1 - \xi$ and $q_I < \xi$.

Proof of Claim 4. Fix $\xi > 0$. Fix an equilibrium in which the competent firm plays according to $\bar{\sigma}^{\text{CP}}$ in the certification phase. Fix $\hat{L}_\xi := \max\{\hat{L}^I_\xi, \hat{L}^C_\xi\}$, where $\hat{L}^I_\xi$ and $\hat{L}^C_\xi$ are chosen below. Fix some $L \geq \hat{L}_\xi$. Similarly, fix some $\hat{N}_\xi$ that is chosen below, and fix some $N \geq \hat{N}_\xi$.

The variable $\hat{L}^I_\xi$ is chosen such that for every $L \geq \hat{L}^I_\xi$, the inept firm's probability of qualifying satisfies $q_I < \xi$. To see that this $\hat{L}^I_\xi$ exists, note that by Chebychev's inequality,

$$q_I \leq \frac{L(1-\rho)\rho}{((1-2\rho)L - L^\beta)^2}. \tag{25}$$

Because the right side tends to 0 as $L \to \infty$, $\hat{L}^I_\xi$ exists.

I next turn to the competent firm's probability of qualifying in this equilibrium. In each period t in the screening phase, because the rating $\varnothing$ is uninformative, the firm's reputation is μ. In the equilibrium, given the competent firm's effort e_t in this period (and the consumer's correct conjecture of this effort), this firm's profit in this period is $\rho + \mu e_t(1-2\rho) - ce_t$ according to (2).

In the equilibrium, let $v_\theta(\delta; Q)$ denote a qualified, type-θ firm's continuation payoff at the beginning of the certification phase, and let $v_\theta(\delta; U)$ denote a unqualified, type-θ firm's continuation payoff at the beginning of the certification phase. The competent firm's payoff in this equilibrium is

$$\begin{aligned} u_C(\delta) = (1-\delta) \sum_{t=0}^{L-1} \delta^t \left(\rho + \mu e_t(1-2\rho) - ce_t\right) \\ + (1-q_C)\delta^L v_C(\delta; U) + q_C \delta^L v_C(\delta; Q). \end{aligned} \tag{26}$$

Let q^*_C denote the implied probability that the competent firm is qualified if it exerts full effort in all periods during the screening phase. The competent firm's payoff from exerting full effort in all periods during the screening phase, when consumers expect that it chooses effort e_t in period t during the screening phase, is

$$\begin{aligned} u_C^*(\delta) = (1-\delta)\sum_{t=0}^{L-1}\delta^t\left(\rho + \mu e_t(1-2\rho) - c\right) \\ + (1-q_C^*)\delta^L v_C(\delta;U) + q_C^*\delta^L v_C(\delta;Q). \end{aligned} \tag{27}$$

Define

$$\bar{v}_C(Q) := \lim_{\delta\to 1} v_C(\delta;Q), \tag{28}$$

$$\bar{v}_C(U) := \lim_{\delta\to 1} v_\theta(\delta;U). \tag{29}$$

Fix $\psi \in (0, \xi[\bar{v}_C(Q) - \bar{v}_C(U)]/2]$. I show in (32) below that, given L and N chosen at the outset of this proof, ψ is well-defined. Let $\hat{L}_\xi^C := \max\{\hat{L}', \hat{L}''\}$, where $\hat{L}'$ and $\hat{L}''$ are chosen below. Then, let $\hat{\delta}_{L,N,\xi} := \max\{\hat{\delta}'_L, \hat{\delta}''_{L,N}, \hat{\delta}'''_\psi\}$, where $\hat{\delta}'_L, \hat{\delta}''_{L,N}, \hat{\delta}'''_\psi \in (0,1)$ are also chosen below, and fix some $\delta \geq \hat{\delta}_{L,N,\xi}$.

The variables $\hat{L}'$ and $\hat{\delta}'_L$ are chosen such that for every $L \geq \hat{L}'$ and for every $\delta \geq \hat{\delta}'_L$,

$$u_C^*(\delta) > \bar{v}_C(Q) - \psi, \tag{30}$$

where $u_C^*(\delta)$ is given in (27) and $\bar{v}_C(Q)$ is given in (28). To see that these $\hat{L}'$ and $\hat{\delta}'_L$ exist, note that by Chebychev's inequality,

$$q_C^* \geq 1 - \frac{L(1-\rho)\rho}{L^{2\beta}} = 1 - \frac{(1-\rho)\rho}{L^{2\beta-1}}. \tag{31}$$

Because $\beta \in (\frac{1}{2}, 1)$, (31) implies that $\lim_{L\to\infty} q_C^* = 1$. Moreover, because

$$\lim_{\delta\to 1} u_C^*(\delta) = (1-q_C^*)\lim_{\delta\to 1} v_C(\delta;U) + q_C^*\lim_{\delta\to 1} v_C(\delta;Q),$$

$$\lim_{q_C^*\to 1}\left[(1-q_C^*)\lim_{\delta\to 1} v_C(\delta;U) + q_C^*\lim_{\delta\to 1} v_C(\delta;Q)\right] = \lim_{\delta\to 1} v_C(\delta;Q) = \bar{v}_C(Q),$$

these $\hat{L}'$ and $\hat{\delta}'_L$ exist.

Next, the variables $\hat{L}''$, $\hat{N}_\xi$, and $\hat{\delta}''_{L,N}$ are chosen such that for every $L \geq \hat{L}''$ and every $N \geq \hat{N}_\xi$, and for every $\delta \geq \hat{\delta}''_{L,N}$,

$$\bar{v}_C(Q) > \bar{v}_C(U), \tag{32}$$

and so ψ is well-defined. To see that these $\hat{L}''$, $\hat{N}_\xi$, and $\hat{\delta}''_{L,N}$ exist, note that by taking the limit as $\delta \to 1$, and then the limit as $L \to \infty$, $\bar{v}_C(Q) = g^N$, where g^N is identified in the proof of Lemma 2. Thus, by using the calculations in the proof of Lemma 2,

$$\begin{aligned}
&\lim_{N\to\infty} \bar{v}_C(Q) \\
=\ &\lim_{N\to\infty} g^N \\
=\ &\begin{cases} 1-\rho-c-\rho\frac{(1-\mu)(1-2\rho)(1-K(1-2\rho))}{1-\rho}, & \text{if } \hat{c} < \frac{(1-\mu)(1-2\rho)^2(1-K(1-2\rho))}{1-\rho}, \\ 1-\rho-c-\rho\frac{\hat{c}}{1-2\rho}, & \text{otherwise.} \end{cases} \\
=\ &f(\hat{c})(1-\rho-c) + (1-f(\hat{c}))(p_\mu - c),
\end{aligned}$$

where

$$f(\hat{c}) = \begin{cases} \frac{(1-2\rho)(1+K\rho)}{1-\rho}, & \text{if } \hat{c} < \frac{(1-\mu)(1-2\rho)^2(1-K(1-2\rho))}{1-\rho}, \\ \frac{1-\rho(\hat{c}-4\rho+4)-\mu(1-2\rho)^2}{(1-\mu)(1-2\rho)^2}, & \text{otherwise,} \end{cases}$$

is the proportion of time during which the competent firm receives trade payoff $1-\rho-c$. Because the rating transition probabilities upon qualification and those upon not qualifying in the certification phase are identical in the equilibrium,

$$\lim_{N\to\infty} \bar{v}_C(U) = f(\hat{c})\rho + (1-f(\hat{c}))p_\mu.$$

By direct calculations,

$$\lim_{N\to\infty} \bar{v}_C(Q) > \lim_{N\to\infty} \bar{v}_C(U),$$

yielding the desired observation in (32).

Finally, the variable $\hat{\delta}'''_\psi$ is chosen such that for every $\delta \geq \hat{\delta}'''_\psi$, $|u_C(\delta) - \lim_{\delta\to 1} u_C(\delta)| \leq$

ψ and therefore, by (26), (28), and (29),

$$u_C(\delta) \leq \psi + (1 - q_C)\bar{v}_C(U) + q_C\bar{v}_C(Q). \tag{33}$$

Suppose now, towards a contradiction, that $q_C \leq 1 - \xi$. Then,

$$\begin{aligned} \psi + \xi\bar{v}_C(U) + (1-\xi)\bar{v}_C(Q) &\geq \psi + (1-q_C)\bar{v}_C(U) + q_C\bar{v}_C(Q) \\ &\geq u_C(\delta) \\ &\geq u_C^*(\delta) \\ &> \bar{v}_C(Q) - \psi, \end{aligned}$$

where the first line follows because $\bar{v}_C(Q) > \bar{v}_C(U)$ and $q_C \leq 1 - \xi$, the second line follows from (33), the third line follows because the competent firm must play its best reply in equilibrium, and the last line follows from (30). This derivation implies that

$$\psi > \frac{\xi}{2}\left[\bar{v}_C(C) - \bar{v}_C(I)\right],$$

yielding a contradiction to the definition of ψ, as desired. ■

I now complete the proof of Lemma 3. I first show that if L and N are sufficiently large, then in any candidate equilibrium in which the consumers expect the competent firm to play according to $\bar{\sigma}^{\text{CP}}$ in the certification phase, this firm has no profitable deviation in that phase. By Lemma 2, if $N \geq N'$ and if $\delta \geq \delta'_N$, then the competent firm has no profitable deviation from its effort choices specified in $\bar{\sigma}^{\text{CP}}$ in the certification phase and its incentive is strict when consumers believe that the competent type is qualified with probability one and the inept type is unqualified with probability one. Thus, by Claim 3 above, there exist $\bar{q}_C$ and $\underline{q}_I$ such that if $q_C > \bar{q}_C$ and $q_I < \underline{q}_I$, then if $N \geq N'$ and $\delta \geq \delta'_N$, the competent firm has no profitable deviation from its effort choices specified in $\bar{\sigma}^{\text{CP}}$ in the certification phase when consumers believe that the competent type is qualified with probability q_C and the inept type is qualified with probability q_I in the equilibrium.

Fix some $\xi_\varepsilon \in (0, \min\{\bar{\xi}, 1 - \sqrt{1-\varepsilon}\})$ where $\bar{\xi} := \max\{1 - \bar{q}_C, \underline{q}_I\}$. Set $\underline{L}_\varepsilon :=$

$\max\{\hat{L}_{\bar{\xi}}, \hat{L}_{\xi_\varepsilon}\}$, and $\underline{N}_\varepsilon := \max\{N', \hat{N}_{\bar{\xi}}, \hat{N}_{\xi_\varepsilon}, \tilde{N}_{\xi_\varepsilon}\}$, where N' is identified in Lemma 2, and $\hat{L}_{\bar{\xi}}$, $\hat{L}_{\xi_\varepsilon}$, $\hat{N}_{\bar{\xi}}$, and $\hat{N}_{\xi_\varepsilon}$ are identified in Claim 4, and $\tilde{N}_{\xi_\varepsilon}$ is chosen below to be sufficiently large. Fix $L \geq \underline{L}_\varepsilon$ and $N \geq \underline{N}_\varepsilon$. Set $\underline{\delta}_{L,N,\varepsilon} := \max\{\hat{\delta}_{L,N,\xi_\varepsilon}, \tilde{\delta}_{\varepsilon,\xi_\varepsilon}\}$, where $\hat{\delta}_{L,N,\xi_\varepsilon}$ is identified in Claim 4 and $\tilde{\delta}_{\varepsilon,\xi_\varepsilon}$ is chosen below to be sufficiently large. Fix $\delta \geq \underline{\delta}_{L,N,\varepsilon}$. By Claim 4, $q_C > 1 - \bar{\xi} \geq \bar{q}_C$, and $q_I < \bar{\xi} \leq \underline{q}_I$, and therefore the competent firm has no profitable deviation from $\bar{\sigma}^{\mathrm{CP}}$ in the certification phase.

Thus, to show that there exists an equilibrium in which the competent firm plays according to $\bar{\sigma}^{\mathrm{CP}}$ in the certification phase, it suffices to show that conditional on this firm playing according to $\bar{\sigma}^{\mathrm{CP}}$ in the certification phase, there exists a strategy profile $(\sigma_t)_{t=0}^{L-1}$ that the consumers conjecture the competent type to play in the screening phase, given which the competent type's best reply in the screening phase is precisely this strategy profile. Let Σ^{SP} denote the set of strategies $\sigma^{\mathrm{SP}} \equiv (\sigma_t)_{t=0}^{L-1}$ from which a competent firm can choose to play in the screening phase, fixing its strategy in the certification phase to be $\bar{\sigma}^{\mathrm{CP}}$. Because its effort choices are defined on $[0, 1]$, Σ^{SP} is compact. Moreover, any strategy profile σ^{SP} that the competent firm plays in the screening phase induces a probability of being qualified, denoted by $q_C(\sigma^{\mathrm{SP}})$. This qualification probability, as well as the inept type's qualification probability q_I, determines the competent firm's continuation payoff in the beginning of the certification phase upon its qualification, denoted by $v_C(\delta; Q, q_C, q_I)$, and its counterpart upon not qualifying, denoted by $v_C(\delta; U, q_C, q_I)$, given consumers' (correct) conjecture that the competent firm plays according to $\bar{\sigma}^{\mathrm{CP}}$ in the certification phase. Because the mapping from effort to outcome in each trade is continuous, $q_C(\sigma^{\mathrm{SP}})$ is continuous. Moreover, because the mapping from the competent firm's qualification probability to consumers' payment is continuous, $v_C(\delta; Q, q_C(\sigma^{\mathrm{SP}}), q_I)$ and $v_C(\delta; U, q_C(\sigma^{\mathrm{SP}}), q_I)$ are continuous in $q_C(\sigma^{\mathrm{SP}})$ and thus in σ^{SP}. Then, given consumers' conjecture that the competent firm plays some $\hat{\sigma}^{\mathrm{SP}} \equiv (\hat{\sigma}_t)_{t=0}^{L-1}$ in the screening phase, define

$$\mathbf{B}(\hat{\sigma}^{\mathrm{SP}}) := \arg\max_{\tilde{\sigma}^{\mathrm{SP}} \in \Sigma^{\mathrm{SP}}} \mathbf{E}_{\tilde{\sigma}^{\mathrm{SP}}} \Bigg[(1-\delta) \sum_{t=0}^{L-1} \delta^t \left(p_t(\vec{r}_t) - ce_t \right) \\ + (1 - q_C(\sigma_S))\delta^L v_C(\delta; U, q_C(\sigma^{\mathrm{SP}}), q_I) + q_C(\sigma^{\mathrm{SP}})\delta^L v_C(\delta; Q, q_C(\sigma^{\mathrm{SP}}), q_I) \Bigg],$$

as the set of the competent firm's best reply in the screening phase, where $\mathbf{E}_{\tilde{\sigma}^{\mathrm{SP}}}[\cdot]$

denotes the expectation operator over histories of play in the screening phase induced by $\tilde{\sigma}^{\mathrm{SP}}$, conditional on the competent firm playing according to σ^{CP} in the certification phase. Given any $\hat{\sigma}^{\mathrm{SP}} \in \Sigma^{\mathrm{SP}}$, $\mathbf{B}(\hat{\sigma}^{\mathrm{SP}})$ is closed, convex and nonempty. Glicksberg (1952)'s fixed-point theorem ensures that there is a fixed point, denoted by $\bar{\sigma}^{\mathrm{SP}}$. Thus, an equilibrium exists in which the competent firm plays $\bar{\sigma}^{\mathrm{SP}}$ in the screening phase and $\bar{\sigma}^{\mathrm{CP}}$ in the certification phase. Let $\bar{\sigma}$ denote this equilibrium.

Finally, I turn to the competent firm's average discounted sum of efforts (4). By Claim 4,

$$q_C > 1 - \xi_\varepsilon > 1 - \bar{\xi} \geq \bar{q}_C, \quad \text{and } q_I < \xi_\varepsilon < \bar{\xi} \leq \underline{q}_I,$$

and so $\bar{\sigma}$ is an equilibrium. In this equilibrium, upon the competent firm's qualification, the specified rating transitions and the fixed strategy profile induce an irreducible and aperiodic Markov chain over $R \setminus \{\varnothing\}$ with a unique stationary distribution. Let $\lambda^Q(r)$ denote the associated probability that rating r realizes in this stationary distribution. Let $P_C[r_t = r^N|Q]$ denote the probability that a qualified competent firm's rating in period t in the certification phase is r^N in this equilibrium. Then

$$\lim_{t\to\infty} P_C[r_t = r^N|Q] = \lambda^Q(r^N) = \frac{(\gamma(1-\rho)-\rho)\left(\frac{\rho}{\gamma(1-\rho)}\right)^N}{\gamma(1-\rho)-\rho\left(\frac{\rho}{\gamma(1-\rho)}\right)^N},$$

which tends to zero as $N \to \infty$.

The variable $\tilde{N}_{\xi_\varepsilon}$ is chosen such that for every $N \geq \tilde{N}_{\xi_\varepsilon}$, there is T_N such that for every $t \geq T_N + L$, the probability that a qualified, competent firm does not obtain rating r^N in period t in the certification phase is $P^C[r_t \neq r^N|Q] > 1 - \xi_\varepsilon$. Set

$$\tilde{\delta}_{\varepsilon,\xi_\varepsilon} := \left(\frac{1-\varepsilon}{(1-\xi_\varepsilon)^2}\right)^{\frac{1}{T_N+L}}.$$

Note that $\tilde{\delta}_{\varepsilon,\xi_\varepsilon} < 1$ by definition of ξ_ε. In this equilibrium, the competent firm's average expected discounted sum of effort (4) is

$$
\begin{aligned}
\mathbf{E}^{P_C}\left[(1-\delta)\sum_{t=0}^{\infty}\delta^t e_t\right] &\geq q_C(1-\delta)\sum_{t=T_N}^{\infty}\delta^t P_C[r_t \neq r^N|Q] \\
&> q_C\delta^{T_N+L}(1-\xi_\varepsilon) \\
&> (1-\xi_\varepsilon)^2\delta^{T_N+L} \\
&\geq (1-\xi_\varepsilon)^2(\tilde{\delta}_{\varepsilon,\xi_\varepsilon})^{T_N+L} \\
&\geq 1-\varepsilon,
\end{aligned}
$$

completing the proof. ■

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